Fluctuations of Casimir forces on finite objects. I. Spheres and hemispheres

1992 J. Phys. A: Math. Gen. 25 3015

(http://iopscience.iop.org/0305-4470/25/10/028)

J. Phys. A: Math. Gen. 25 (1992) 3015–3037. Printed in the UK

Fluctuations of Casimir forces on finite objects: I. Spheres and hemispheres

Claudia Eberlein

School of Mathematical and Physical Sciences, University of Sussex, Brighton BN1 9QH, UK

Received 24 October 1991

Abstract. The mean-square forces that result from the zero-point fluctuations of quantized fields are calculated when acting on spheres and hemispheres of variable sizes. For the Maxwell field the boundary conditions of a perfectly conducting surface are imposed; the scalar field is investigated for Neumann and Dirichlet boundary conditions. The force is averaged over a finite time T; small and large objects are distinguished on the scale of cT. The results for the sphere and the hemisphere are compared with those for a piston that is embedded in an infinite plane. A small hemisphere and a small piston are found to have fluctuations of the same order of magnitude, while on a small sphere the fluctuations are by two orders of magnitude smaller because of correlations of fluctuations on the two sides of the sphere. Large spheres are shown to fit into the picture of large objects being composed of many patches, each with the fluctuations impinging as on a large piston.

1. Introduction and outline

While, for a rather long time, Casimir forces on perfectly conducting surfaces (Casimir 1948) used to be discussed without referring to the details of their fluctuating nature, the fluctuations themselves recently came into consideration (Barton 1991a). Their investigation by means of zero-temperature quantum field theory aims to disclose their basic features as well as to give some estimates of their order of magnitude under conditions close to those in experiments.

In contrast to the work previously done by Barton (1991a, b) where, for the purpose of exploring general properties of fluctuations of Casimir forces, almost exclusively a piston embedded in an infinite conducting plane is considered, the present and the following paper (Eberlein 1992) centre on the investigation of these fluctuations on objects of finite extent, especially spheres, hemispheres, and spheroids. These will be compared with the calculations for an embedded piston, captivating in their simplicity. However, an experiment necessarily involves a finite apparatus, which gives rise to questions concerning the influence of this finiteness on the fluctuations. Therefore, this article continues and complements the preceding article by Barton (1991a) where the finiteness of a real experimental arrangement is taken into account only by averaging the Casimir stress over a certain area out of an infinite plane. On a really finite object this, inevitably, would misestimate the fluctuations; most crucially due to the neglect of the correlations on the two sides of the object, which is unavoidable in that approach and must significantly overestimate the fluctuations.

0305-4470/92/103015+23$04.50

Since these errors increase by orders of magnitude for objects small compared to a typical correlation length, the investigation of fluctuations on isolated finite objects presented here proves essential for the understanding of the nature of the fluctuations and for reliable estimations of their magnitude.

A suitable way for studying Casimir forces is by use of the stress tensor $\mathbf{S}$ which is given in terms of the space-like components of the stress–energy tensor; i.e. for the Maxwell field by†

$$S_{ij} = \frac{1}{4\pi}\left[E_i E_j + B_i B_j - \delta_{ij}\tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2)\right] \tag{1.1}$$

and for a scalar field by

$$S_{ij} = \nabla_i\psi\nabla_j\psi + \delta_{ij}\tfrac{1}{2}\left[\left(\frac{\partial\psi}{\partial t}\right)^2 - (\nabla\psi)^2\right] \tag{1.2}$$

in an orthonormal base. The appropriate boundary conditions on the surface of the object under consideration will simplify the above expressions later on. In any case the boundary conditions will be taken as time-independent, i.e. the object under consideration is assumed to be at rest and to gain only a negligible velocity during the measurement.

The mean-square deviation of the stress is defined as usual

$$\Delta\mathbf{S}^2 = \langle 0 \mid \mathbf{S}^2 \mid 0\rangle - \langle 0 \mid \mathbf{S} \mid 0\rangle^2 .$$

After insertion of a complete set of two-particle eigenstates this reads

$$\Delta\mathbf{S}^2 = \tfrac{1}{2}\sum_{\lambda,\lambda'} \mid \langle\lambda,\lambda' \mid \mathbf{S} \mid 0\rangle \mid^2 \tag{1.3}$$

where λ abbreviates all quantum numbers necessary to specify the photon state, i.e. the wave vector and the polarization.

Apart from averaging the stress $\mathbf{S}$ over a finite surface area the experimental situation suggests an averaging over a certain time T since measurements are not made instantaneously at a fixed time‡. Therefore, time-averaging is introduced by

$$\overline{\mathbf{S}} = \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\mathbf{S}(t) \tag{1.4}$$

where the function $f(t)$ (with $f(t) \geqslant 0$, $\int_{-\infty}^{\infty}\mathrm{d}t f(t) = 1$) is thought of as being significantly different from zero only in an interval $2T$ during a measurement. The actual shape of $f(t)$ proves of minor importance (see Barton 1991a), so that a commitment further on to Lorentzian time-averaging

$$f(t) = \frac{T/\pi}{t^2 + T^2}$$

† CGS units are used; $\hbar = c = 1$ is understood throughout if not explicitly indicated. All special functions are defined as by Gradshteyn and Ryshik (1980, GR for short).

‡ $\Delta\mathbf{S}^2$ will turn out to be divergent without time-averaging as one would expect from the physics.

does not alter basic results†. It is merely chosen for its technical handiness since the calculation will use its Fourier transform which is simply

$$\int_{-\infty}^{\infty} \mathrm{d}t \, \frac{T/\pi}{t^2 + T^2} \mathrm{e}^{\mathrm{i}\omega t} = \mathrm{e}^{-|\omega| T}. \tag{1.5}$$

The idea of the calculation is to expand the Maxwell or scalar field, respectively, into its normal modes and to quantize it by introducing annihilation and creation operators as usual in the scheme of the canonical quantization. Inserting the normal-mode expansions into the formulae for the stress tensor **S** (given by (1.1) and (1.2), respectively) and noticing that the only non-vanishing matrix-elements contributing in (1.3) are those where two creation operators are acting on the vacuum

$$\langle \lambda, \lambda' \mid a_\lambda^\dagger a_{\lambda'}^\dagger \mid 0 \rangle = 1 \tag{1.6}$$

one can evaluate $\Delta \mathbf{S}^2$ by picking up the coefficients of $a_\lambda^\dagger a_{\lambda'}^\dagger$ in the expression for **S** and integrating their squared modulus according to (1.3).

This will be done for the Maxwell field in section 2, for the scalar field with Neumann boundary conditions in section 3 and with Dirichlet boundary conditions in section 4. For each of these fields the stress will be integrated over a sphere and a hemisphere to obtain the respective forces on these objects in a fixed direction along the z-axis. For the scalar fields the perpendicular forces on a piston embedded in an infinite plane are calculated in appendix A, since they are important for comparison and have not yet been given anywhere else. Appendix B contributes some non-trivial technical details necessary for the reproducibility of the calculations. The main results are summarized in tables 1–3.

The far more difficult case of a flat circular disk as a limiting case of a spheroid with vanishing eccentricity is tackled in the following paper (Eberlein 1992).

2. The Maxwell field

On the surface of a perfectly conducting body the Maxwell field is required to fulfil the boundary conditions that the electric field tangential to the surface and the magnetic field perpendicular to it vanish. For a perfectly conducting sphere this means in spherical coordinates

$$E_\theta = E_\varphi = B_r = 0. \tag{2.1}$$

The stress tensor (1.1) simplifies to

$$\begin{aligned} S_{rr} &= \frac{1}{8\pi} \left(E_r^2 - B_\theta^2 - B_\varphi^2 \right) \\ S_{\theta r} &= S_{\varphi r} = 0. \end{aligned} \tag{2.2}$$

Accordingly, the z-component of the total force acting on a sphere of radius R is given by

$$F_z = \frac{R^2}{8\pi} \int \mathrm{d}\Omega \left(E_r^2 - B_\theta^2 - B_\varphi^2 \right) \cos\theta. \tag{2.3}$$

† For different time-averaging functions the results for $\Delta \mathbf{S}^2$ show the same power dependence on T, but have different numerical coefficients in front.

2.1. The TE and TM normal modes

The quantized electric and magnetic fields

$$E = \sum_{\sigma=1,2} \int \mathrm{d}^3k\, \mathrm{i}\omega \left(a_{k,\sigma} A(k,\sigma) \mathrm{e}^{-\mathrm{i}\omega t} - a^{\dagger}_{k,\sigma} A^{*}(k,\sigma) \mathrm{e}^{\mathrm{i}\omega t} \right)$$

$$B = \sum_{\sigma=1,2} \int \mathrm{d}^3k\, \nabla \times \left(a_{k,\sigma} A(k,\sigma) \mathrm{e}^{-\mathrm{i}\omega t} + a^{\dagger}_{k,\sigma} A^{*}(k,\sigma) \mathrm{e}^{\mathrm{i}\omega t} \right)$$

involve the normal modes $A(k,\sigma)$ of the vector potential in Coulomb gauge which are solutions of the Helmholtz equation

$$\nabla^2 A + k^2 A = 0. \tag{2.4}$$

To find such solutions it is convenient to proceed from the well-known scalar solution

$$\Phi = \frac{4\pi}{(2\pi)^{3/2}} \sum_{\ell,m} \mathrm{e}^{-\mathrm{i}\delta_\ell}\, \mathrm{i}^\ell \left[\cos\delta_\ell\, j_\ell(kr) + \sin\delta_\ell\, y_\ell(kr)\right] Y_\ell^{m*}(\hat{k}) Y_\ell^m(\hat{r}) \tag{2.5}$$

which is normalized to behave like $\mathrm{e}^{\mathrm{i}k\cdot r}/(2\pi)^{3/2}$ for $kr \to \infty$. The phase δ_ℓ will be chosen to meet the required boundary conditions on the surface of the sphere $r = R$.

The vector field operators invariant under rotation are r, ∇, $L = -\mathrm{i}r \times \nabla$ and $\nabla \times L$ (see Brink and Satchler 1961). Since r fails to commute with ∇^2, only the last three operators can be used to generate vector solutions of the Helmholtz equation. However, $\nabla\Phi$ is an irrotational field†, so that the solenoidal fields‡ $L\Phi$ and $(\nabla \times L)\Phi$ have to be employed for representations of the electromagnetic field. Choosing $A_{(1)} \sim L\Phi$ and $A_{(2)} \sim 1/k\, \nabla \times L\Phi$ and watching the correct normalization§ one obtains

$$A_{(1)\theta} = \frac{2}{\sqrt{\omega}} \sum_{\ell,m} \frac{\mathrm{e}^{-\mathrm{i}\delta_\ell}\, \mathrm{i}^\ell}{\sqrt{\ell(\ell+1)}} \left[\cos\delta_\ell\, j_\ell(kr) + \sin\delta_\ell\, y_\ell(kr)\right] Y_\ell^{m*}(\hat{k}) \frac{(-m)}{\sin\theta} Y_\ell^m(\hat{r})$$

$$A_{(1)\varphi} = \frac{2}{\sqrt{\omega}} \sum_{\ell,m} \frac{\mathrm{e}^{-\mathrm{i}\delta_\ell}\, \mathrm{i}^\ell}{\sqrt{\ell(\ell+1)}} \left[\cos\delta_\ell\, j_\ell(kr) + \sin\delta_\ell\, y_\ell(kr)\right] Y_\ell^{m*}(\hat{k}) (-\mathrm{i}) \frac{\partial Y_\ell^m(\hat{r})}{\partial\theta}$$

$$A_{(1)r} = 0$$

† This means $\nabla \times \nabla\Phi = 0$. Furthermore, a choice $A \sim \nabla\Phi$ would be in contradiction with the Coulomb gauge since $\nabla \cdot \nabla\Phi = -k^2\Phi \neq 0$.

‡ This means $\nabla \cdot L\Phi = 0$ and $\nabla \cdot (\nabla \times L)\Phi = 0$ as required in the Coulomb gauge.

§ The normalization is fixed by equating the total energy of the electromagnetic field $H_{\mathrm{em}} = (1/8\pi) \int \mathrm{d}^3r\, (E^2 + B^2)$ and the energy of a set of harmonic oscillators $H_{\mathrm{osc}} = \int \mathrm{d}^3k\, (\omega/2)\, (a_k^{\dagger} a_k + a_k a_k^{\dagger})$ for each polarization.

$$
\begin{aligned}
A_{(2)\theta} &= \frac{2}{\sqrt{\omega}} \sum_{\ell,m} \frac{\mathrm{e}^{-\mathrm{i}\delta_\ell}\,\mathrm{i}^\ell}{\sqrt{\ell(\ell+1)}} \frac{1}{kr} \left[kr\cos\delta_\ell\, j_\ell(kr) + kr\sin\delta_\ell\, y_\ell(kr)\right]' \\
&\quad \times Y_\ell^{m*}(\hat{\boldsymbol{k}})\,\mathrm{i}\,\frac{\partial Y_\ell^m(\hat{\boldsymbol{r}})}{\partial\theta} \\
A_{(2)\varphi} &= \frac{2}{\sqrt{\omega}} \sum_{\ell,m} \frac{\mathrm{e}^{-\mathrm{i}\delta_\ell}\,\mathrm{i}^\ell}{\sqrt{\ell(\ell+1)}} \frac{1}{kr} \left[kr\cos\delta_\ell\, j_\ell(kr) + kr\sin\delta_\ell\, y_\ell(kr)\right]' \\
&\quad \times Y_\ell^{m*}(\hat{\boldsymbol{k}})\,\frac{(-m)}{\sin\theta}\,Y_\ell^m(\hat{\boldsymbol{r}}) \\
A_{(2)r} &= \frac{2}{\sqrt{\omega}} \sum_{\ell,m} \frac{\mathrm{e}^{-\mathrm{i}\delta_\ell}\,\mathrm{i}^\ell}{\sqrt{\ell(\ell+1)}} \frac{\mathrm{i}\ell(\ell+1)}{kr} \left[\cos\delta_\ell\, j_\ell(kr)\right. \\
&\quad \left. +\sin\delta_\ell\, y_\ell(kr)\right] Y_\ell^{m*}(\hat{\boldsymbol{k}})Y_\ell^m(\hat{\boldsymbol{r}}).
\end{aligned}
\tag{2.6}
$$

These are essentially the transverse electric (TE) and the transverse magnetic (TM) polarizations of the Hansen multipole fields (Biedenharn and Louck 1989). They do not directly correspond to the standard basis of vector spherical harmonics (see Newton 1982) but are convenient combinations of them. In terms of the vector fields (2.6) the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields read (see also Ruppin 1982)

$$
\begin{aligned}
\boldsymbol{E}_{\mathrm{TE}} &= \int \mathrm{d}^3\boldsymbol{k}\,\mathrm{i}\omega\left[a_k^{\mathrm{TE}}\boldsymbol{A}_{(1)}^{\mathrm{TE}}\mathrm{e}^{-\mathrm{i}\omega t} - \mathrm{HC}\right] \\
\boldsymbol{B}_{\mathrm{TE}} &= \int \mathrm{d}^3\boldsymbol{k}\,k\left[a_k^{\mathrm{TE}}\boldsymbol{A}_{(2)}^{\mathrm{TE}}\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{HC}\right] \\
\boldsymbol{E}_{\mathrm{TM}} &= \int \mathrm{d}^3\boldsymbol{k}\,\mathrm{i}\omega\left[a_k^{\mathrm{TM}}\boldsymbol{A}_{(2)}^{\mathrm{TM}}\mathrm{e}^{-\mathrm{i}\omega t} - \mathrm{HC}\right] \\
\boldsymbol{B}_{\mathrm{TM}} &= \int \mathrm{d}^3\boldsymbol{k}\,k\left[a_k^{\mathrm{TM}}\boldsymbol{A}_{(1)}^{\mathrm{TM}}\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{HC}\right].
\end{aligned}
\tag{2.7}
$$

In order to obtain the amplitudes $\boldsymbol{A}^{\mathrm{TE,TM}}$ from (2.6) the phase δ_ℓ has to be replaced by the phases $\delta_\ell^{\mathrm{TE,TM}}$ which, according to the boundary conditions (2.1), are determined by

$$
\begin{aligned}
\tan\delta_\ell^{\mathrm{TE}} &= -\frac{j_\ell(kR)}{y_\ell(kR)} \\
\tan\delta_\ell^{\mathrm{TM}} &= -\frac{[kRj_\ell(kR)]'}{[kRy_\ell(kR)]'}.
\end{aligned}
\tag{2.8}
$$

Here and in the following a prime behind a bracket denotes a differentiation with respect to the argument of the Bessel function. (Not to be confused with primes that serve for distinguishing different variables.)

Now the matrix elements relevant for calculating the mean-square deviation of the force (2.3) can be read off the modes by use of (1.6),

$$
\langle k_{\mathrm{TE}}, k'_{\mathrm{TE}} \mid S_{rr}\cos\theta \mid 0\rangle = -\frac{1}{\pi}\sqrt{\omega\omega'}\,\mathrm{e}^{\mathrm{i}(\omega+\omega')t} \sum_{\ell,m}\sum_{\ell',m'} \frac{\mathrm{e}^{\mathrm{i}\delta_\ell^{\mathrm{TE}}}(-\mathrm{i})^\ell}{\sqrt{\ell(\ell+1)}} \frac{\mathrm{e}^{\mathrm{i}\delta_{\ell'}^{\mathrm{TE}}}(-\mathrm{i})^{\ell'}}{\sqrt{\ell'(\ell'+1)}}
$$

$$\times \frac{1}{kk'R^2}\left[kR\cos\delta_\ell^{\mathrm{TE}} j_\ell(kR) + kR\sin\delta_\ell^{\mathrm{TE}} y_\ell(kR)\right]'$$
$$\times \left[k'R\cos\delta_{\ell'}^{\mathrm{TE}} j_{\ell'}(k'R) + k'R\sin\delta_{\ell'}^{\mathrm{TE}} y_{\ell'}(k'R)\right]' Y_\ell^m(\hat{\boldsymbol{k}})Y_{\ell'}^{m'}(\hat{\boldsymbol{k}}')$$
$$\times \left[\frac{mm'}{\sin^2\theta}Y_\ell^{m*}(\hat{\boldsymbol{r}})Y_{\ell'}^{m'*}(\hat{\boldsymbol{r}}) - \frac{\partial Y_\ell^{m*}(\hat{\boldsymbol{r}})}{\partial\theta}\frac{\partial Y_{\ell'}^{m'*}(\hat{\boldsymbol{r}})}{\partial\theta}\right]\cos\theta$$

$$\langle \boldsymbol{k}_{\mathrm{TM}}, \boldsymbol{k}'_{\mathrm{TM}} \mid S_{rr}\cos\theta \mid 0\rangle = -\frac{1}{\pi}\sqrt{\omega\omega'}\,\mathrm{e}^{\mathrm{i}(\omega+\omega')t}\sum_{\ell,m}\sum_{\ell',m'}\frac{\mathrm{e}^{\mathrm{i}\delta_\ell^{\mathrm{TE}}}(-\mathrm{i})^\ell}{\sqrt{\ell(\ell+1)}}\frac{\mathrm{e}^{\mathrm{i}\delta_{\ell'}^{\mathrm{TE}}}(-\mathrm{i})^{\ell'}}{\sqrt{\ell'(\ell'+1)}}$$
$$\times \left[\cos\delta_\ell^{\mathrm{TM}} j_\ell(kR) + \sin\delta_\ell^{\mathrm{TM}} y_\ell(kR)\right]$$
$$\times \left[\cos\delta_{\ell'}^{\mathrm{TM}} j_{\ell'}(k'R) + \sin\delta_{\ell'}^{\mathrm{TM}} y_{\ell'}(k'R)\right] Y_\ell^m(\hat{\boldsymbol{k}})Y_{\ell'}^{m'}(\hat{\boldsymbol{k}}')$$
$$\times \left[\frac{mm'}{\sin^2\theta}Y_\ell^{m*}(\hat{\boldsymbol{r}})Y_{\ell'}^{m'*}(\hat{\boldsymbol{r}}) - \frac{\partial Y_\ell^{m*}(\hat{\boldsymbol{r}})}{\partial\theta}\frac{\partial Y_{\ell'}^{m'*}(\hat{\boldsymbol{r}})}{\partial\theta}\right.$$
$$\left. - \frac{\ell(\ell+1)}{kR}\frac{\ell'(\ell'+1)}{k'R}Y_\ell^{m*}(\hat{\boldsymbol{r}})Y_{\ell'}^{m'*}(\hat{\boldsymbol{r}})\right]\cos\theta$$

$$\langle \boldsymbol{k}_{\mathrm{TE}}, \boldsymbol{k}'_{\mathrm{TM}} \mid S_{rr}\cos\theta \mid 0\rangle = -\frac{1}{\pi}\sqrt{\omega\omega'}\,\mathrm{e}^{\mathrm{i}(\omega+\omega')t}\sum_{\ell,m}\sum_{\ell',m'}\frac{\mathrm{e}^{\mathrm{i}\delta_\ell^{\mathrm{TE}}}(-\mathrm{i})^\ell}{\sqrt{\ell(\ell+1)}}\frac{\mathrm{e}^{\mathrm{i}\delta_{\ell'}^{\mathrm{TE}}}(-\mathrm{i})^{\ell'}}{\sqrt{\ell'(\ell'+1)}}$$
$$\times \frac{1}{kR}\left[kR\cos\delta_\ell^{\mathrm{TE}} j_\ell(kR) + kR\sin\delta_\ell^{\mathrm{TE}} y_\ell(kR)\right]'$$
$$\times \left[\cos\delta_{\ell'}^{\mathrm{TM}} j_{\ell'}(k'R) + \sin\delta_{\ell'}^{\mathrm{TM}} y_{\ell'}(k'R)\right] Y_\ell^m(\hat{\boldsymbol{k}})Y_{\ell'}^{m'}(\hat{\boldsymbol{k}}')$$
$$\times \left[(-\mathrm{i})\frac{\partial Y_\ell^{m*}(\hat{\boldsymbol{r}})}{\partial\theta}\frac{-m'}{\sin\theta}Y_{\ell'}^{m'*}(\hat{\boldsymbol{r}}) + \frac{-m}{\sin\theta}Y_\ell^{m*}(\hat{\boldsymbol{r}})\,\mathrm{i}\,\frac{\partial Y_{\ell'}^{m'*}(\hat{\boldsymbol{r}})}{\partial\theta}\right]\cos\theta.$$

The integration over the sphere then involves the angular integrals $\Im_1$, $\Im_2$ and $\Im_3$ which are evaluated in appendix B.1 (see (B.1), (B.2) and (B.3), respectively). Using the results (B.5), (B.6) and (B.4) for the surface integration, and taking into account that time-averaging according to (1.4) simply changes the harmonic time dependence of the above matrix elements into multiplication with the Fourier transform of the time-averaging function (given in (1.5) for the Lorentzian average function employed here), one finds for the time-averaged matrix elements of the force F_z:

$$\langle \boldsymbol{k}_{\mathrm{TE}}, \boldsymbol{k}'_{\mathrm{TE}} \mid \overline{F_z} \mid 0\rangle = -\frac{1}{\pi}\sqrt{\omega\omega'}\,R^2\,\mathrm{e}^{-(\omega+\omega')T}\sum_{\ell,m}\frac{\mathrm{e}^{\mathrm{i}\delta_\ell^{\mathrm{TE}}}(-\mathrm{i})^\ell}{\sqrt{\ell(\ell+1)}}\frac{\mathrm{e}^{\mathrm{i}\delta_{\ell-1}^{\mathrm{TE}}}(-\mathrm{i})^{\ell-1}}{\sqrt{\ell(\ell-1)}}$$
$$\times\left\{\frac{1}{kk'R^2}\left[kR\cos\delta_\ell^{\mathrm{TE}} j_\ell(kR) + kR\sin\delta_\ell^{\mathrm{TE}} y_\ell(kR)\right]'\right.$$
$$\times\left[k'R\cos\delta_{\ell-1}^{\mathrm{TE}} j_{\ell-1}(k'R) + k'R\sin\delta_{\ell-1}^{\mathrm{TE}} y_{\ell-1}(k'R)\right]'$$
$$\times Y_\ell^m(\hat{\boldsymbol{k}})Y_{\ell-1}^{-m}(\hat{\boldsymbol{k}}')$$
$$+\frac{1}{kk'R^2}\left[kR\cos\delta_{\ell-1}^{\mathrm{TE}} j_{\ell-1}(kR) + kR\sin\delta_{\ell-1}^{\mathrm{TE}} y_{\ell-1}(kR)\right]'$$
$$\left.\times\left[k'R\cos\delta_\ell^{\mathrm{TE}} j_\ell(k'R) + k'R\sin\delta_\ell^{\mathrm{TE}} y_\ell(k'R)\right]' Y_{\ell-1}^{-m}(\hat{\boldsymbol{k}})Y_\ell^m(\hat{\boldsymbol{k}}')\right\}$$
$$\times(-1)^{m+1}(\ell^2-1)\sqrt{\frac{(\ell-m)(\ell+m)}{(2\ell+1)(2\ell-1)}} \tag{2.9}$$

$$\langle \boldsymbol{k}_{\mathrm{TM}}, \boldsymbol{k}'_{\mathrm{TM}} \mid \overline{F_z} \mid 0\rangle = -\frac{1}{\pi}\sqrt{\omega\omega'}\, R^2\, \mathrm{e}^{-(\omega+\omega')T} \sum_{\ell,m} \frac{\mathrm{e}^{\mathrm{i}\delta_\ell^{\mathrm{TE}}}(-\mathrm{i})^\ell}{\sqrt{\ell(\ell+1)}} \frac{\mathrm{e}^{\mathrm{i}\delta_{\ell-1}^{\mathrm{TE}}}(-\mathrm{i})^{\ell-1}}{\sqrt{\ell(\ell-1)}}$$
$$\times \left\{\left[\cos\delta_\ell^{\mathrm{TM}} j_\ell(kR) + \sin\delta_\ell^{\mathrm{TM}} y_\ell(kR)\right]\right.$$
$$\times \left[\cos\delta_{\ell-1}^{\mathrm{TM}} j_{\ell-1}(k'R) + \sin\delta_{\ell-1}^{\mathrm{TM}} y_{\ell-1}(k'R)\right]\, Y_\ell^m(\hat{\boldsymbol{k}}) Y_{\ell-1}^{-m}(\hat{\boldsymbol{k}}')$$
$$+ \left[\cos\delta_{\ell-1}^{\mathrm{TM}} j_{\ell-1}(kR) + \sin\delta_{\ell-1}^{\mathrm{TM}} y_{\ell-1}(kR)\right]$$
$$\left.\times \left[\cos\delta_\ell^{\mathrm{TM}} j_\ell(k'R) + \sin\delta_\ell^{\mathrm{TM}} y_\ell(k'R)\right]\, Y_{\ell-1}^{-m}(\hat{\boldsymbol{k}}) Y_\ell^m(\hat{\boldsymbol{k}}')\right\}$$
$$\times (-1)^{m+1}(\ell^2-1)\left(1+\frac{\ell'^2}{kk'R^2}\right)\sqrt{\frac{(\ell-m)(\ell+m)}{(2\ell+1)(2\ell-1)}} \tag{2.10}$$

$$\langle \boldsymbol{k}_{\mathrm{TE}}, \boldsymbol{k}'_{\mathrm{TM}} \mid \overline{F_z} \mid 0\rangle = -\frac{1}{\pi}\sqrt{\omega\omega'}\, R^2\, \mathrm{e}^{-(\omega+\omega')T} \sum_{\ell,m} \frac{\mathrm{e}^{\mathrm{i}\delta_\ell^{\mathrm{TE}}+\mathrm{i}\delta_\ell^{\mathrm{TM}}}}{\ell(\ell+1)}(-1)^\ell$$
$$\times \frac{1}{kR}\left[kR\cos\delta_\ell^{\mathrm{TE}} j_\ell(kR) + kR\sin\delta_\ell^{\mathrm{TE}} y_\ell(kR)\right]'$$
$$\times \left[\cos\delta_\ell^{\mathrm{TM}} j_\ell(k'R) + \sin\delta_\ell^{\mathrm{TM}} y_\ell(k'R)\right]$$
$$\times Y_\ell^m(\hat{\boldsymbol{k}}) Y_\ell^{-m}(\hat{\boldsymbol{k}}')\,(-1)^{m+1}\mathrm{i}m. \tag{2.11}$$

The summation in (1.3) now means an integration $\int \mathrm{d}^3\boldsymbol{k} \int \mathrm{d}^3\boldsymbol{k}'$ and a subsequent summation over polarizations. Having the summations $\sum_{m=-\ell}^{\ell} = 2\ell+1$ and $\sum_{m=-\ell}^{\ell} m^2 = 1/3\,\ell(\ell+1)(2\ell+1)$ already executed, one arrives at

$$\Delta\overline{F_z}^2 = \frac{1}{3\pi^2R^4}\sum_{\ell=1}^{\infty}\frac{1}{\ell(\ell+1)}\{(\ell-1)(\ell+1)^2[\mathcal{D}_1(\ell)\mathcal{D}_1(\ell-1)$$
$$+\mathcal{D}_2(\ell)\mathcal{D}_2(\ell-1) + 2\ell^2\mathcal{D}_3(\ell)\mathcal{D}_3(\ell-1) + \ell^4\mathcal{D}_4(\ell)\mathcal{D}_4(\ell-1)]$$
$$+(2\ell+1)\mathcal{D}_1(\ell)\mathcal{D}_2(\ell)\} \tag{2.12}$$

which involves the integrals

$$\mathcal{D}_1(\ell) = \int_0^\infty \mathrm{d}x\,\mathrm{e}^{-\lambda x}\, x\,\left\{\cos\delta_\ell^{\mathrm{TE}}\,[xj_\ell(x)]' + \sin\delta_\ell^{\mathrm{TE}}\,[xy_\ell(x)]'\right\}^2$$
$$\mathcal{D}_2(\ell) = \int_0^\infty \mathrm{d}x\,\mathrm{e}^{-\lambda x}\, x^3\left\{\cos\delta_\ell^{\mathrm{TM}} j_\ell(x) + \sin\delta_\ell^{\mathrm{TM}} y_\ell(x)\right\}^2,$$
$$\mathcal{D}_3(\ell) = \int_0^\infty \mathrm{d}x\,\mathrm{e}^{-\lambda x}\, x^2\left\{\cos\delta_\ell^{\mathrm{TM}} j_\ell(x) + \sin\delta_\ell^{\mathrm{TM}} y_\ell(x)\right\}^2$$
$$\mathcal{D}_4(\ell) = \int_0^\infty \mathrm{d}x\,\mathrm{e}^{-\lambda x}\, x\,\left\{\cos\delta_\ell^{\mathrm{TM}} j_\ell(x) + \sin\delta_\ell^{\mathrm{TM}} y_\ell(x)\right\}^2.$$

The dimensionless variable x replaces the former kR, and

$$\lambda = \frac{2T}{R} \tag{2.13}$$

is dimensionless too. Inserting the expressions for the phases (2.8) and using the Wronskian of the spherical Bessel functions (Abramowitz and Stegun 1964 (AS for short in the following) 10.1.6)

$$j_\ell(x)y'_\ell(x) - j'_\ell(x)y_\ell(x) = \frac{1}{x^2}$$

one may rewrite the above integrals

$$\mathcal{D}_1(\ell) = \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-\lambda x}\, \frac{1}{x}\, \frac{1}{j_\ell^2(x) + y_\ell^2(x)} \tag{2.14}$$

$$\mathcal{D}_2(\ell) = \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-\lambda x}\, x\, \frac{1}{\{[x j_\ell(x)]'\}^2 + \{[x y_\ell(x)]'\}^2} \tag{2.15}$$

$$\mathcal{D}_3(\ell) = \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-\lambda x}\, \frac{1}{\{[x j_\ell(x)]'\}^2 + \{[x y_\ell(x)]'\}^2} \tag{2.16}$$

$$\mathcal{D}_4(\ell) = \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-\lambda x}\, \frac{1}{x}\, \frac{1}{\{[x j_\ell(x)]'\}^2 + \{[x y_\ell(x)]'\}^2}. \tag{2.17}$$

The denominator in the integrand of $\mathcal{D}_1(\ell)$ is easily recognized as a polynomial in inverse powers of $(2x)$

$$j_\ell^2(x) + y_\ell^2(x) = \frac{1}{x^2} \sum_{k=0}^{\ell} \frac{(\ell+k)!(2k)!}{(\ell-k)!(k!)^2} (2x)^{-2k} \tag{2.18}$$

(see AS 10.1.27). One would expect that the denominator occurring in $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$ can be expressed by a similar polynomial. Indeed, using the moduli, M and N, of the Hankel functions and their derivatives, respectively, as defined in AS (9.2.19, 20 and 22), one finds after a little algebra

$$\{[x j_\ell(x)]'\}^2 + \{[x y_\ell(x)]'\}^2 = \frac{\pi}{2x} \left(\tfrac{1}{4} M_{\ell+1/2}^2 + x\, M_{\ell+1/2} M_{\ell+1/2}' + x^2 N_{\ell+1/2}^2 \right)$$

which eventually yields (see AS 10.1.27, 9.2.30)

$$= 1 + \sum_{k=0}^{\ell} \frac{(\ell+k)!(2k)!}{(\ell-k)!k!(k+1)!} (2x)^{-2k-2}\, 2\, [k(k+1)(2k+1) - \ell(\ell+1)]\,. \tag{2.19}$$

It has to be pointed out that the two polynomials, (2.18) and (2.19), are by no means approximations, but still exact expressions. In the scattering theory of electromagnetic waves the quantity $[x^2(j_\ell^2 + y_\ell^2)]^{-1} \equiv A_\ell^{-1}$ is known as the penetrability of the angular momentum barrier (Biedenharn and Louck 1989), and the quantity $B_\ell \equiv [(x j_\ell)']^2 + [(x y_\ell)']^2$, occurring in the denominators of (2.15) to (2.17), can be shown to be closely related to it via

$$B_\ell = \left[1 - \frac{\ell(\ell+1)}{x^2}\right] A_\ell + \frac{1}{2} A_\ell''.$$

However, although the polynomials derived above seem to simplify the integrals (2.14) to (2.17) considerably, general analytic solutions for them can hardly be found. It is reasonable to investigate the two limiting cases where the typical duration of time-averaging T is much larger (or smaller) than the geometrical dimensions of the object under consideration, i.e. e.g. the radius R of the sphere.

For $T \gg R$ ($\lambda \to \infty$ in (2.13)) modes with high frequencies (i.e. short wavelengths) are averaged away; so this is a long-wavelengths limit since the latter deliver the dominant contribution to the fluctuations. Conversely, for $T \ll R$ ($\lambda \to 0$ in (2.13)) the frequency cut-off $1/T$ is large; so this clearly is the short-wavelengths limit.

2.2. The small sphere ($R \ll T$)

In the long-wavelengths limit, where λ is very large, the exponentials $e^{-\lambda x}$ impose a strong damping on all powers of x. Thus, by Watson's lemma, one derives, proceeding from (2.18) and (2.19),

$$\mathcal{D}_1(\ell) = \frac{(2\ell+1)}{4}\frac{(\ell!)^2}{(2\ell)!}\left(\frac{R}{T}\right)^{2\ell+2}\left[1+\mathrm{O}\left(\frac{R^2}{T^2}\right)\right] \tag{2.20}$$

$$\mathcal{D}_2(\ell) = \frac{(2\ell+3)(2\ell+1)(\ell+1)}{8}\frac{[(\ell-1)!]^2}{(2\ell)!}\left(\frac{R}{T}\right)^{2\ell+4}\left[1+\mathrm{O}\left(\frac{R^2}{T^2}\right)\right] \tag{2.21}$$

$$\mathcal{D}_3(\ell) = \frac{(2\ell+1)(\ell+1)}{4}\frac{[(\ell-1)!]^2}{(2\ell)!}\left(\frac{R}{T}\right)^{2\ell+3}\left[1+\mathrm{O}\left(\frac{R^2}{T^2}\right)\right] \tag{2.22}$$

$$\mathcal{D}_4(\ell) = \frac{(2\ell+1)}{4}\frac{[(\ell-1)!]^2}{(2\ell)!}\left(\frac{R}{T}\right)^{2\ell+2}\left[1+\mathrm{O}\left(\frac{R^2}{T^2}\right)\right]. \tag{2.23}$$

Inserted in (2.12) this gives in leading order

$$\Delta\overline{F_z}^2 \sim \frac{35\,R^2}{2^6\pi^4 T^{10}}\left(\pi R^2\right)^2. \tag{2.24}$$

2.3. The large sphere ($R \gg T$)

In the short-wavelengths limit the calculation of the leading term is less straightforward. However, the problem is closely related to that of high-frequency scattering by white spheres (see Mott and Massey 1965, section II.6), so that one may get some inspiration about the methods to be used from there.

Since $1/\lambda$ tends to infinity now, the cut-off is very large, i.e. contributions from large values of x dominate in the integrals (2.14) to (2.17). Likewise the summation over ℓ in (2.12) gets its dominant contributions now from high ℓ. Employing Debye's asymptotic expansions for Bessel functions with large values of the indices (AS 9.3.7, 8, 15, 16) one can, after a few algebraic steps, rewrite

$$\mathcal{D}_1 = \nu^2\int_0^1 \mathrm{d}z\,\mathrm{e}^{-\lambda\nu z}\sqrt{1-z^2}\,\mathrm{e}^{2\nu(\sqrt{1-z^2}-\mathrm{arcsech}\,z)} + \nu^2\int_1^\infty \mathrm{d}z\,\mathrm{e}^{-\lambda\nu z}\sqrt{z^2-1}$$

$$\mathcal{D}_2 = \nu^2\int_0^1 \mathrm{d}z\,\mathrm{e}^{-\lambda\nu z}\frac{z^2}{\sqrt{1-z^2}}\mathrm{e}^{2\nu(\sqrt{1-z^2}-\mathrm{arcsech}\,z)} + \nu^2\int_1^\infty \mathrm{d}z\,\mathrm{e}^{-\lambda\nu z}\frac{z^2}{\sqrt{z^2-1}}$$

$$\mathcal{D}_3 = \nu^2\int_0^1 \mathrm{d}z\,\mathrm{e}^{-\lambda\nu z}\frac{z}{\sqrt{1-z^2}}\mathrm{e}^{2\nu(\sqrt{1-z^2}-\mathrm{arcsech}\,z)} + \nu^2\int_1^\infty \mathrm{d}z\,\mathrm{e}^{-\lambda\nu z}\frac{z}{\sqrt{z^2-1}}$$

$$\mathcal{D}_4 = \nu^2\int_0^1 \mathrm{d}z\,\mathrm{e}^{-\lambda\nu z}\frac{1}{\sqrt{1-z^2}}\mathrm{e}^{2\nu(\sqrt{1-z^2}-\mathrm{arcsech}\,z)} + \nu^2\int_1^\infty \mathrm{d}z\,\mathrm{e}^{-\lambda\nu z}\frac{1}{\sqrt{z^2-1}}$$

where $\nu \equiv \ell + 1/2$ and z replaces x/ν.

In order to evaluate the leading term of (2.12) the summation over ℓ is turned into an integration

$$\Delta\overline{F_z}^2 = \frac{1}{3\pi^2R^4}\int_0^\infty \mathrm{d}\nu\Big\{\nu\,[\mathcal{D}_1(\nu)]^2 + \nu\,[\mathcal{D}_2(\nu)]^2 + \nu^3\,[\mathcal{D}_3(\nu)]^2 + \nu^5\,[\mathcal{D}_4(\nu)]^2 + \frac{1}{\nu}\mathcal{D}_1(\nu)\mathcal{D}_2(\nu)\Big\}. \tag{2.25}$$

Interchanging the z and the ν integration (they are all convergent), i.e. performing the one over ν first, leads to

$$\int_0^\infty \mathrm{d}\nu\,\nu\,[\mathcal{D}_1(\nu)]^2 = \frac{5!}{\lambda^6}\int_1^\infty \mathrm{d}z\int_1^\infty \mathrm{d}\tilde{z}\,\frac{\sqrt{z^2-1}\sqrt{\tilde{z}^2-1}}{(z+\tilde{z})^6} + \frac{1}{\lambda^6}\,\mathrm{O}(\lambda)$$

$$\int_0^\infty \mathrm{d}\nu\,\nu\,[\mathcal{D}_2(\nu)]^2 = \frac{5!}{\lambda^6}\int_1^\infty \mathrm{d}z\int_1^\infty \mathrm{d}\tilde{z}\,\frac{z^2\tilde{z}^2}{\sqrt{z^2-1}\sqrt{\tilde{z}^2-1}}\,\frac{1}{(z+\tilde{z})^6} + \frac{1}{\lambda^6}\,\mathrm{O}\left(\lambda^{1/3}\right)$$

$$\int_0^\infty \mathrm{d}\nu\,\nu^3\,[\mathcal{D}_3(\nu)]^2 = \frac{5!}{\lambda^6}\int_1^\infty \mathrm{d}z\int_1^\infty \mathrm{d}\tilde{z}\,\frac{z\tilde{z}}{\sqrt{z^2-1}\sqrt{\tilde{z}^2-1}}\,\frac{1}{(z+\tilde{z})^6} + \frac{1}{\lambda^6}\,\mathrm{O}\left(\lambda^{1/3}\right)$$

$$\int_0^\infty \mathrm{d}\nu\,\nu^5\,[\mathcal{D}_4(\nu)]^2 = \frac{5!}{\lambda^6}\int_1^\infty \mathrm{d}z\int_1^\infty \mathrm{d}\tilde{z}\,\frac{1}{\sqrt{z^2-1}\sqrt{\tilde{z}^2-1}}\,\frac{1}{(z+\tilde{z})^6} + \frac{1}{\lambda^6}\,\mathrm{O}\left(\lambda^{1/3}\right)$$

$$\int_0^\infty \mathrm{d}\nu\,\frac{1}{\nu}\mathcal{D}_1(\nu)\mathcal{D}_2(\nu) = \mathrm{O}\left(\frac{1}{\lambda^4}\right)$$

where it can be recognized that obviously values of x with $x \geqslant \ell + \frac{1}{2}$ are giving the leading contributions in the integrals. At first glance one might be a little surprised by the occurence of an expansion in one-third powers. This, however, is known as a peculiarity of high-frequency scattering by spheres and circular cylinders (see Wu 1956), and is, mathematically speaking, a consequence of the uniform asymptotic expansions of the Bessel functions (Langer's formulae, Erdélyi *et al* 1953, 7.13.4.) where Bessel functions with indices $\pm\frac{1}{3}$ crop up.

The double integrals to be evaluated as the coefficients of the leading power $1/\lambda^6$ can be quite tricky if not gone about in the right way. Even then it is a rather tedious business, so that their evaluation is shifted to appendix B.2. With (B.7) to (B.10) taken into account (2.25) gives the final result in the short-wavelengths limit

$$\Delta\overline{F_z}^2 \sim \frac{1}{3\times 2^3\pi^4 R^2 T^6}\,\left(\pi R^2\right)^2 . \tag{2.26}$$

2.4. The normal stress on a sphere

With the formalism already introduced in the preceding subsection the calculation of the mean normal stress $\langle S_{rr}\rangle$, acting from outside on the surface of a perfectly conducting sphere, lies at hand. From (2.2) and (2.7) one gets, by simply picking up the coefficients of $a_k a_k^\dagger$,

$$\langle S_{rr}\rangle = \frac{1}{8\pi}\int \mathrm{d}^3\boldsymbol{k}k^2\left(|\,A^{\mathrm{TE}}_{(1)r}\,|^2 + |\,A^{\mathrm{TM}}_{(2)r}\,|^2 - |\,A^{\mathrm{TM}}_{(1)\theta}\,|^2 - |\,A^{\mathrm{TM}}_{(1)\varphi}\,|^2 - |\,A^{\mathrm{TE}}_{(2)\theta}\,|^2 - |\,A^{\mathrm{TE}}_{(2)\varphi}\,|^2\right)$$

and with (2.6)

$$
\begin{aligned}
\langle S_{rr}\rangle = \frac{1}{2\pi}\int_0^\infty \mathrm{d}k\, k^3 \sum_{\ell,m} \Bigg\{ & \frac{\ell(\ell+1)}{k^2R^2} Y_\ell^{m*}(\hat{r}) Y_\ell^m(\hat{r}) \\
& \times \left[\cos\delta_\ell^{\mathrm{TM}} j_\ell(kR) + \sin\delta_\ell^{\mathrm{TM}} y_\ell(kR)\right]^2 \\
& - \left(\frac{m^2}{\sin^2\theta} Y_\ell^{m*}(\hat{r}) Y_\ell^m(\hat{r}) + \frac{\partial Y_\ell^{m*}(\hat{r})}{\partial\theta}\frac{\partial Y_\ell^m(\hat{r})}{\partial\theta}\right) \\
& \times \frac{1}{\ell(\ell+1)} \left[\cos\delta_\ell^{\mathrm{TM}} j_\ell(kR) + \sin\delta_\ell^{\mathrm{TM}} y_\ell(kR)\right]^2 \\
& - \left(\frac{m^2}{\sin^2\theta} Y_\ell^{m*}(\hat{r}) Y_\ell^m(\hat{r}) + \frac{\partial Y_\ell^{m*}(\hat{r})}{\partial\theta}\frac{\partial Y_\ell^m(\hat{r})}{\partial\theta}\right) \\
& \times \frac{1}{\ell(\ell+1)}\,\frac{1}{k^2R^2}\left(\left[kR\,\cos\delta_\ell^{\mathrm{TE}} j_\ell(kR)\right.\right. \\
& \left.\left. +\; kR\,\sin\delta_\ell^{\mathrm{TE}} y_\ell(kR)\right]'\right)^2 \Bigg\}\,.
\end{aligned}
$$

Of course the normal stress cannot depend on the position $\hat{r}$. For the first term in the above equation this is indeed easily seen by recalling the sum rule for spherical harmonics (Brink and Satchler 1961, appendix iv)

$$
\sum_{m=-\ell}^{\ell} \mid Y_\ell^m \mid^2 = \frac{2\ell+1}{4\pi}.
$$

For the second and the third term one writes

$$
\begin{aligned}
& \sum_{m=-\ell}^{\ell}\left(\frac{m^2}{\sin^2\theta} Y_\ell^{m*} Y_\ell^m + \frac{\partial Y_\ell^{m*}}{\partial\theta}\frac{\partial Y_\ell^m}{\partial\theta}\right) \\
& \qquad = \sum_{m=-\ell}^{\ell} (LY_\ell^m)^* (LY_\ell^m) = \ell(\ell+1)\sum_{m=-\ell}^{\ell} Y_\ell^{m*} Y_\ell^m.
\end{aligned}
$$

The last step follows after decomposing L in L_+, L_- and L_z and taking into account their action on the Ys. Then

$$
\begin{aligned}
\langle S_{rr}\rangle = \frac{1}{8\pi^2}\int_0^\infty \mathrm{d}k\, k^3 \sum_{\ell=1}^{\infty}(2\ell+1)\Bigg\{ & \left(\frac{\ell(\ell+1)}{k^2R^2} - 1\right) \\
& \times \left[\cos\delta_\ell^{\mathrm{TM}} j_\ell(kR) + \sin\delta_\ell^{\mathrm{TM}} y_\ell(kR)\right]^2 \\
& - \frac{1}{k^2R^2}\left(\left[kR\,\cos\delta_\ell^{\mathrm{TE}} j_\ell(kR) + kR\,\sin\delta_\ell^{\mathrm{TE}} y_\ell(kR)\right]'\right)^2\Bigg\}
\end{aligned}
$$

which is clearly divergent. Therefore a cut-off frequency Ω is introduced by a factor $\mathrm{e}^{-k/\Omega}$. Since the cut-off is to be removed at the end of the calculation (i.e. $\Omega \to \infty$) the same arguments as for the evaluation of $\Delta\overline{F_z}^2$ in the short-wavelengths limit apply. In the same notation as before one finds, after a few lines,

$$
\langle S_{rr}\rangle = \frac{1}{4\pi^2}\int_0^\infty \mathrm{d}\nu\left[\nu^3\mathcal{D}_4(\nu) - \nu\mathcal{D}_2(\nu) - \nu\mathcal{D}_1(\nu)\right]\,.
$$

Note that Ω^{-1} replaces $2T$, and therefore $\lambda = 1/(\Omega R)$. With the $\mathcal{D}$s approximated as above and the ν integration performed first, the remaining (one-dimensional) integration is trivial and entails

$$\langle S_{rr} \rangle = -\frac{\Omega^4}{\pi^2} .$$

This is no surprise but merely a repetition of the result for the normal stress on a piston (Barton 1991a). Both are necessarily the same since the leading order terms arise from volume contributions, and the surface- (and curvature-) dependent terms are of next-to-leading order Ω^2 only (Balian and Duplantier 1977). However, as the present approach proves inconvenient for calculations beyond the leading order this is skipped here.

2.5. The small hemisphere

Next, the fluctuations on a small hemisphere, with the polar axis in z direction, are evaluated as this, compared with the result for the sphere, will give a clue about the influence of correlations between the two sides of an object.

The matrix elements $\langle \lambda, \lambda' \mid S_{rr} \cos\theta \mid 0 \rangle$ are now to be integrated over the surface of a hemisphere only, i.e. $\int \mathrm{d}\Omega \to \int \mathrm{d}\Omega_{\mathrm{H}} = \int_0^{\pi/2} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \sin\theta$. Although, in principle the integrals $\Im_1$, $\Im_2$ and $\Im_3$ (see appendix B.1) could, even for a hemisphere, be evaluated in terms of general formulae (see Shabde 1937 for integrals over products of two associated Legendre polynomials), this turns out to give rather unpleasant infinite sums†. A much quicker way is to calculate the hemisphere-integrals $\Im_1$, $\Im_2$ and $\Im_3$ for ℓ and ℓ' up to 2 by hand. Proceeding as before for the small sphere one finds

$$\begin{aligned}\Delta\overline{F_z}^2 = \frac{1}{2\pi^2 R^4}\bigg\{ & \frac{99}{512}[\mathcal{D}_1(1)]^2 + \tfrac{1}{4}\mathcal{D}_1(1)\mathcal{D}_1(2) + \tfrac{475}{2304}[\mathcal{D}_1(2)]^2 + \tfrac{99}{512}[\mathcal{D}_2(1)]^2 \\ & + \tfrac{45}{64}[\mathcal{D}_3(1)]^2 + \tfrac{27}{32}[\mathcal{D}_4(1)]^2 + \tfrac{1}{4}\mathcal{D}_2(1)\mathcal{D}_2(2) \\ & + 2\mathcal{D}_3(1)\mathcal{D}_3(2) + 4\mathcal{D}_4(1)\mathcal{D}_4(2) \\ & + \tfrac{475}{2304}[\mathcal{D}_2(2)]^2 + \tfrac{175}{64}[\mathcal{D}_3(2)]^2 + \tfrac{1575}{128}[\mathcal{D}_4(2)]^2 + \tfrac{1}{4}\mathcal{D}_1(1)\mathcal{D}_2(1) \\ & + \tfrac{15}{256}\mathcal{D}_1(1)\mathcal{D}_2(2) + \tfrac{15}{256}\mathcal{D}_1(2)\mathcal{D}_2(1) + \tfrac{5}{36}\mathcal{D}_1(2)\mathcal{D}_2(2) + \cdots \bigg\}\end{aligned}$$

and with (2.20) to (2.23) eventually

$$\Delta\overline{F_z}^2 \sim \left[\frac{3^4 \times 59}{2^{16}\pi^4 T^8} + \frac{5 \times 137 R^2}{2^{11}\pi^4 T^{10}} + \mathrm{O}\left(\frac{R^4}{T^{12}}\right)\right](\pi R^2)^2 . \tag{2.27}$$

† This can be understood immediately once one has noticed that the simple structure of the results (B.5) and (B.6) for the sphere originates from the selection rule for dipole matrix elements $\Delta\ell = \pm 1$ which, however, does not hold on a hemisphere.

3. The scalar field with Neumann boundary conditions

A scalar field may be thought of as a representation of an acoustic field, i.e. as the velocity potential $\boldsymbol{v} = -\nabla\psi$ in a fluid. Small vibrations of a non-viscous, compressible fluid are then governed by the wave equation. If these compressional waves are impinging on an impenetrable obstacle the boundary condition on its surface (having the normal vector $\hat{\boldsymbol{n}}$) is $\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}\psi = 0$. On a sphere this means

$$\boldsymbol{\nabla}_r \psi = 0\,. \tag{3.1}$$

Then the stress-tensor components (1.2) read

$$\begin{aligned} S_{rr} &= \tfrac{1}{2}(\partial\psi/\partial t)^2 - \tfrac{1}{2}(\boldsymbol{\nabla}_\theta\psi)^2 - \tfrac{1}{2}(\boldsymbol{\nabla}_\varphi\psi)^2 \\ S_{\theta r} &= S_{\varphi r} \;=\; 0. \end{aligned} \tag{3.2}$$

3.1. The normal modes

Quantizing the scalar field according to

$$\psi = \int \mathrm{d}^3\boldsymbol{k}\,\left(u_k\,\psi(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\omega t} + u_k^\dagger\,\psi^*(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\omega t}\right) \tag{3.3}$$

one quickly writes down the normal modes (see (2.5) apart from normalization):

$$\begin{aligned} \frac{\partial\psi}{\partial t} &= \frac{4\pi}{(2\pi)^{3/2}}(-\mathrm{i}\omega)\sum_{\ell,m}\frac{\mathrm{e}^{-\mathrm{i}\delta_\ell}\,\mathrm{i}^\ell}{\sqrt{2k}}\left[\cos\delta_\ell\, j_\ell(kr) + \sin\delta_\ell\, y_\ell(kr)\right] Y_\ell^{m*}(\hat{\boldsymbol{k}})Y_\ell^m(\hat{\boldsymbol{r}}) \\ \nabla_r\psi &= \frac{4\pi}{(2\pi)^{3/2}}\,k\sum_{\ell,m}\frac{\mathrm{e}^{-\mathrm{i}\delta_\ell}\,\mathrm{i}^\ell}{\sqrt{2k}}\left[\cos\delta_\ell\, j'_\ell(kr) + \sin\delta_\ell\, y'_\ell(kr)\right] Y_\ell^{m*}(\hat{\boldsymbol{k}})Y_\ell^m(\hat{\boldsymbol{r}}) \\ \nabla_\theta\psi &= \frac{4\pi}{(2\pi)^{3/2}}\sum_{\ell,m}\frac{\mathrm{e}^{-\mathrm{i}\delta_\ell}\,\mathrm{i}^\ell}{\sqrt{2k}}\left[\cos\delta_\ell\, j_\ell(kr) + \sin\delta_\ell\, y_\ell(kr)\right] Y_\ell^{m*}(\hat{\boldsymbol{k}})\frac{1}{r}\frac{\partial Y_\ell^m(\hat{\boldsymbol{r}})}{\partial\theta} \\ \nabla_\varphi\psi &= \frac{4\pi}{(2\pi)^{3/2}}\sum_{\ell,m}\frac{\mathrm{e}^{-\mathrm{i}\delta_\ell}\,\mathrm{i}^\ell}{\sqrt{2k}}\left[\cos\delta_\ell\, j_\ell(kr) + \sin\delta_\ell\, y_\ell(kr)\right] \\ &\quad\times Y_\ell^{m*}(\hat{\boldsymbol{k}})\frac{1}{r\sin\theta}\frac{\partial Y_\ell^m(\hat{\boldsymbol{r}})}{\partial\varphi}. \end{aligned} \tag{3.4}$$

Note that $\partial\psi/\partial t$ and $\nabla_r\psi$ include $\ell = 0$ contributions which are not admitted in the electromagnetic modes (2.6) since electromagnetic waves are transverse. The boundary condition (3.1) fixes the phase δ_ℓ via

$$\tan\delta_\ell = -\frac{j'_\ell(kR)}{y'_\ell(kR)}\,. \tag{3.5}$$

On similar lines as before for the electromagnetic field the matrix element of the stress (3.2) is found to be

$$
\begin{aligned}
\langle k, k' \mid S_{rr} \cos\theta \mid 0\rangle = \frac{1}{\pi} \sum_{\ell,m} \sum_{\ell',m'} \frac{\mathrm{e}^{\mathrm{i}\delta_\ell + \mathrm{i}\delta_{\ell'}} (-\mathrm{i})^{\ell+\ell'}}{\sqrt{kk'}} \mathrm{e}^{\mathrm{i}(\omega+\omega')t} \\
\times \left[\cos\delta_\ell\, j_\ell(kR) + \sin\delta_\ell\, y_\ell(kR)\right] \left[\cos\delta_{\ell'}\, j_{\ell'}(k'R)\right. \\
\left. + \sin\delta_{\ell'}\, y_{\ell'}(k'R)\right] Y_\ell^m(\hat{k}) Y_{\ell'}^{m'}(\hat{k}') \left\{ -\omega\omega' Y_\ell^{m*}(\hat{r}) \cos\theta\, Y_{\ell'}^{m'*}(\hat{r}) \right. \\
\left. + \cos\theta \left[\frac{\partial Y_\ell^{m*}(\hat{r})}{\partial\theta} \frac{\partial Y_{\ell'}^{m'*}(\hat{r})}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial Y_\ell^{m*}(\hat{r})}{\partial\varphi} \frac{\partial Y_{\ell'}^{m'*}(\hat{r})}{\partial\varphi} \right] \right\}.
\end{aligned}
$$

The integration of this matrix element over the sphere then employs the integral $\Im_1$ (see (B.2) and (B.6)) and the second term of the integral $\Im_2$ (see (B.1) and (B.5)) and leads, after Lorentzian time-averaging, squaring and subsequent summation over the phonon states as indicated in (1.3), to

$$
\begin{aligned}
\Delta \overline{F_z}^2 = \frac{1}{3\pi^2 R^4} \sum_{\ell=1}^{\infty} \ell \left[\mathcal{P}_1(\ell)\mathcal{P}_1(\ell-1) + 2(\ell^2-1)\mathcal{P}_2(\ell)\mathcal{P}_2(\ell-1)\right. \\
\left. + (\ell^2-1)^2 \mathcal{P}_3(\ell)\mathcal{P}_3(\ell-1)\right]
\end{aligned}
$$

with the $\mathcal{P}$s being

$$
\begin{aligned}
\mathcal{P}_1(\ell) &= \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-\lambda x} \frac{1}{x} \frac{1}{[j_\ell'(x)]^2 + [y_\ell'(x)]^2} \\
\mathcal{P}_2(\ell) &= \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-\lambda x} \frac{1}{x^2} \frac{1}{[j_\ell'(x)]^2 + [y_\ell'(x)]^2} \\
\mathcal{P}_3(\ell) &= \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-\lambda x} \frac{1}{x^3} \frac{1}{[j_\ell'(x)]^2 + [y_\ell'(x)]^2}
\end{aligned}
$$

in the same dimensionless variables $x = kR$ and $\lambda = 2T/R$ as before.

3.2. *The small sphere ($R \ll T$)*

Applying again Watson's Lemma and taking into account the expansion of the spherical Bessel functions for small arguments (AS 10.1.2, 3) one finds for the leading-order term

$$
\Delta \overline{F_z}^2 \sim \frac{5 \times 11 R^2}{3 \times 2^8 \pi^4 T^{10}} \left(\pi R^2\right)^2 . \tag{3.7}
$$

3.3. *The large sphere ($R \gg T$)*

As before for the $\mathcal{D}$s one uses Debye's asymptotic expansions to find an asymptotic expansion for the $\mathcal{P}$s in the limit of $\lambda \to 0$. It turns out that the expansion for $\mathcal{P}_1$ is equal to that for $\mathcal{D}_2$ already given above, and the same applies to the expansions for $\mathcal{P}_2$ and $\mathcal{D}_3$, and to those for $\mathcal{P}_3$ and $\mathcal{D}_4$. Note that the coincidence of the leading terms in the short-wavelengths expansions does not mean that the integrals equal each other.

Turning the summation over ℓ in (3.6) into an integration and performing this integration first, one arrives, after a little algebra, along the same lines as before, and with the help of the integrals evaluated in appendix B.2, at

$$\Delta \overline{F_z}^2 \sim \frac{11}{3^2 \times 2^5 \pi^4 R^2 T^6} \left(\pi R^2\right)^2 . \tag{3.8}$$

3.4. The small hemisphere

For the evaluation of the fluctuations on a small hemisphere the integration of the matrix element $\langle k, k' \mid S_{rr} \cos\theta \mid 0\rangle$ over the hemisphere surface is again most quickly done by hand. One is led to

$$\Delta \overline{F_z}^2 = \frac{1}{2\pi^2 R^4} \Big\{ [\mathcal{P}_1(0)]^2 + \tfrac{1}{6} \mathcal{P}_1(0)\mathcal{P}_1(1) + \tfrac{27}{128} [\mathcal{P}_1(1)]^2 + \tfrac{45}{64} [\mathcal{P}_2(1)]^2 \\ + \tfrac{99}{128} [\mathcal{P}_3(1)]^2 + \dots \Big\}$$

which gives, by use of the large-λ-asymptotics of the $\mathcal{P}$s,

$$\Delta \overline{F_z}^2 \sim \left[\frac{3^2 \times 19 \times 113}{2^{18} \pi^4 T^8} + \frac{7^2 \times 173 R^2}{3^2 \times 2^{15} \pi^4 T^{10}} + \mathrm{O}\left(\frac{R^4}{T^{12}}\right) \right] \left(\pi R^2\right)^2 . \tag{3.9}$$

4. The scalar field with Dirichlet boundary conditions

Although there seems to be no immediate physical application of a scalar field with Dirichlet boundary conditions, i.e. with

$$\psi = 0$$

on the surface of the object under consideration, the investigation of this case will be briefly outlined here for its somewhat surprising results.

The stress-tensor components (1.2) are now reduced to

$$S_{rr} = \tfrac{1}{2} \left(\nabla_r \psi\right)^2 ,$$
$$S_{\theta r} = S_{\varphi r} = 0 .$$

The normal modes are already given in (3.4); the different boundary conditions now entail

$$\tan \delta_\ell = -\frac{j_\ell(kR)}{y_\ell(kR)} .$$

The matrix element of the stress reads

$$\langle \boldsymbol{k}, \boldsymbol{k}' \mid S_{rr} \cos\theta \mid 0\rangle = \frac{1}{\pi} \sum_{\ell,m} \sum_{\ell',m'} \frac{\mathrm{e}^{\mathrm{i}\delta_\ell + \mathrm{i}\delta_{\ell'}} (-\mathrm{i})^{\ell+\ell'}}{\sqrt{kk'}} \mathrm{e}^{\mathrm{i}(\omega+\omega')t} \\ \times [\cos\delta_\ell \, j_\ell(kR) + \sin\delta_\ell \, y_\ell(kR)] \\ \times [\cos\delta_{\ell'} \, j_{\ell'}(k'R) + \sin\delta_{\ell'} \, y_{\ell'}(k'R)] \, Y_\ell^m(\hat{\boldsymbol{k}}) Y_{\ell'}^{m'}(\hat{\boldsymbol{k}}') \\ \times Y_\ell^{m*}(\hat{\boldsymbol{r}}) \cos\theta \, Y_{\ell'}^{m'*}(\hat{\boldsymbol{r}}) .$$

For the surface integration one needs only the second term of the integral $\Im_2$ (see (B.2) and (B.6)). Eventually it is found that

$$\Delta\overline{F_z}^2 = \frac{1}{3\pi^2 R^4}\sum_{\ell=1}^{\infty}\ell\mathcal{D}_1(\ell)\mathcal{D}_1(\ell-1) \tag{4.1}$$

where $\mathcal{D}_1(\ell)$ is given by (2.14).

Along the same lines as before one obtains for the small sphere ($R \ll T$)

$$\Delta\overline{F_z}^2 \sim \frac{1}{2^5\pi^4 R^2 T^6}\left(\pi R^2\right)^2 \tag{4.2}$$

for the large sphere

$$\Delta\overline{F_z}^2 \sim \frac{1}{3^2\times 2^5\pi^4 R^2 T^6}\left(\pi R^2\right)^2 \tag{4.3}$$

and for the hemisphere

$$\Delta\overline{F_z}^2 \sim \left[\frac{1}{2^9\pi^4 R^4 T^4}+\frac{1}{2^7\pi^4 R^2 T^6}+\mathrm{O}\left(\frac{1}{T^8}\right)\right]\left(\pi R^2\right)^2 . \tag{4.4}$$

5. Comparison and conclusions

The results for the Maxwell field are summarized in table 1, those for the scalar field with Neumann and Dirichlet boundary conditions in tables 2 and 3, respectively.

Table 1. The mean square deviations $\Delta\overline{F_z}^2/A^2$ ($A = \pi R^2$ or a^2, respectively) for the Maxwell field.

	Sphere	Hemisphere	Piston†
$a,\ R \ll cT$	$\frac{35R^2}{2^6\pi^4 T^{10}}$	$\frac{3^4\times 59}{2^{16}\pi^4 T^8}$	$\frac{3}{2^7\pi^4 T^8}$
	$\left(1.8\times 10^{-3}\frac{\pi R^2}{T^{10}}\right)$	$\left(7.5\times 10^{-4}\frac{1}{T^8}\right)$	$\left(2.4\times 10^{-4}\frac{1}{T^8}\right)$
$a,\ R \gg cT$	$\frac{1}{3\times 2^3\pi^4 R^2 T^6}$	$\frac{1}{2}\times$ (sphere)	$\frac{1}{2^5\pi^3 a^2 T^6}$
	$\left(1.3\times 10^{-3}\frac{1}{\pi R^2 T^6}\right)$		$\left(1.0\times 10^{-3}\frac{1}{a^2 T^6}\right)$

†These results are taken from the paper by Barton (1991b).

Comparing the piston and the hemisphere with the sphere one sees that the correlations of the fluctuations on the two sides of an object play a very important role for objects that are small on the scale of the time of averaging T; the correlations decrease the fluctuations by two powers of T/R. Judging from the Maxwell field and the scalar field with Neumann boundary conditions one would be inclined to regard piston and hemisphere as roughly the same with respect to the fluctuations, their

Table 2. The mean square deviations $\Delta\overline{F_z}^2/A^2$ ($A = \pi R^2$ or a^2, respectively) for the scalar field with Neumann boundary conditions.

	Sphere	Hemisphere	Piston
$a,\ R \ll cT$	$\dfrac{5\times 11R^2}{3\times 2^8\pi^4 T^{10}}$ $\left(2.3\times 10^{-4}\dfrac{\pi R^2}{T^{10}}\right)$	$\dfrac{3^2\times 19\times 113}{2^{18}\pi^4 T^8}$ $\left(7.6\times 10^{-4}\dfrac{1}{T^8}\right)$	$\dfrac{11}{2^9\pi^4 T^8}$ $\left(2.2\times 10^{-4}\dfrac{1}{T^8}\right)$
$a,\ R \gg cT$	$\dfrac{11}{3\times 2^5\pi^4 R^2 T^6}$ $\left(1.2\times 10^{-3}\dfrac{1}{\pi R^2 T^6}\right)$	$\frac{1}{2}\times$ (sphere)	$\dfrac{11}{3\times 2^7\pi^3 a^2 T^6}$ $\left(0.9\times 10^{-3}\dfrac{1}{a^2 T^6}\right)$

Table 3. The mean square deviations $\Delta\overline{F_z}^2/A^2$ ($A = \pi R^2$ or a^2, respectively) for the scalar field with Dirichlet boundary conditions.

	Sphere	Hemisphere	Piston
$a,\ R \ll cT$	$\dfrac{1}{2^5\pi^4 R^2 T^6}$ $\left(1.0\times 10^{-3}\dfrac{1}{\pi R^2 T^6}\right)$	$\dfrac{1}{2^9\pi^4 R^4 T^4}$ $\left(2.0\times 10^{-4}\dfrac{1}{\pi R^4 T^4}\right)$	$\dfrac{1}{2^9\pi^4 T^8}$ $\left(2.0\times 10^{-5}\dfrac{1}{T^8}\right)$
$a,\ R \gg cT$	$\dfrac{1}{3^2\times 2^5\pi^4 R^2 T^6}$ $\left(1.1\times 10^{-4}\dfrac{1}{\pi R^2 T^6}\right)$	$\frac{1}{2}\times$ (sphere)	$\dfrac{1}{3\times 2^7\pi^3 a^2 T^6}$ $\left(8.4\times 10^{-5}\dfrac{1}{a^2 T^6}\right)$

different geometry bringing along only a numerical factor of around three; both seem to be an equally good model for a one-sided object. The peculiarity of the scalar field with Dirichlet boundary conditions, where this kind of interpretation goes wrong, will be discussed below.

For objects large compared with the duration of time-averaging T it seems plausible that the fluctuations on differently shaped bodies differ only by numerical factors of order one since the correlation length is much smaller than the overall extent of the object and correlations between different sides are therefore of negligible influence.

Furthermore, large bodies can be thought of as being composed of many patches which may be regarded as flat as long as the curvature of the surface under consideration stays reasonably small. Integrating over flat patches that are assumed to have the same fluctuations impinging as there are on a piston, one finds that the fluctuations on a sphere are connected with those on a piston via

$$\frac{\Delta\overline{F_z}^2(\text{sphere})}{\pi R^2} = \frac{4}{3}\,\frac{\Delta\overline{F_z}^2(\text{piston})}{a^2}$$

which is indeed backed up by the results given in tables 1–3. That is why it is pointless to calculate the fluctuations on a large hemisphere from scratch; the result will, of course, be half of that for the sphere. Supporting the above statements with some mathematical rigour, however, requires a consideration of the correlation function of the stress (see Barton 1991b for that).

The comparison of the results for the different fields shows that the Maxwell field and the scalar field with Neumann boundary conditions are remarkably close. One might have expected a much larger difference between the results for the small sphere on account of the different structure of the normal modes, the scalar field admitting $\ell = 0$ contributions and the Maxwell field not.

The results for the fluctuations of the scalar field with Dirichlet boundary conditions on small objects seem not to fit into the picture. Hemisphere and piston have different dependence on T; most oddly the force fluctuations on a hemisphere are independent of its radius. About the last mentioned phenomenon, however, one should not rack one's brain too much since the small hemisphere was a made-up model for comparison with the piston, neither of them being realizable in practice because of their artificial one-sideness. Furthermore, one is reminded of the fact that the natural counterpart of the Maxwell field is the scalar field with Neumann boundary conditions, while the present type of investigation for Dirichlet boundary conditions has no immediate significance in physics.

Of course, one will ask about fluctuations from inside too once one deals with hollow objects. For small objects these fluctuations are exponentially small, i.e. proportional to $e^{-4\omega_0 T}$ (for Lorentzian time averaging) with ω_0 being the lowest eigenmode admitted inside the cavity, as one can convince oneself quickly by performing the calculation for a rectangular box. For small cavities the fluctuations from inside are therefore negligible. For large objects the fluctuations from inside may be calculated in the same way as those from outside, by breaking up the surface into patches.

Things become far more complicated for two objects close together, as, for instance, the set-up of the two parallel plates in the classic Casimir experiment. For this one might construct a model of a sphere that is cut into two hemispheres with a small gap in between. It is not clear whether the slit may be handled as if it were a closed cavity, i.e. whether the density of the modes lengthwise the slit at low frequencies brings about an exponential damping. Judging from scattering experiments in such open-ended cavities where strong resonances occur at frequencies that correspond to the length of the cavity, one would assume that the density of the modes is mainly governed by these resonances and shows a gap-like behaviour at low frequencies, which would make the fluctuations from inside negligible in comparison with those from outside. However, a mathematical back-up of this reasoning involves considerable additional effort and is beyond the scope of the present paper.

Thinking about a direct experimental detection of the fluctuations of Casimir forces one conveniently employs some simple considerations proceeding from the uncertainty relation in order to get general limitations for the measurability. Since the uncertainty in position must not reach the typical correlation length cT, for the model of an effectively immobile object staying valid, the uncertainty in momentum is at least $\hbar/cT$. The impulse acquired by the object during the measurement from the zero-point fluctuations of the field is measurable of course only if it exceeds this momentum uncertainty. Translating all this into formulae (see Barton 1991b, appendix B, for the piston in the Maxwell field), one quickly realizes that, for the Maxwell field as well as for the scalar field with Neumann boundary conditions, the fluctuations on small objects are excluded from being measurable. So, apart from numerical values being small or large measurability is limited, as a matter of principle, to large objects.

Acknowledgment

The author greatly appreciates numerous valuable discussions with Gabriel Barton without whose continuous encouragement this work would not have been done.

Appendix A. Fluctuations of scalar fields on a piston

A1. Neumann boundary conditions

Let the boundary plane be described by $z = 0$ in Cartesian coordinates. The scalar field†

$$\psi = \frac{1}{(2\pi)^{3/2}} \int_0^\infty \mathrm{d}\ell \int \mathrm{d}^2 \boldsymbol{k}_\parallel \sqrt{\frac{2}{\omega}} \cos \ell z \left(u_k \mathrm{e}^{\mathrm{i}\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_\parallel - \mathrm{i}\omega t} + u_k^\dagger \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\parallel \cdot \boldsymbol{r}_\parallel + \mathrm{i}\omega t} \right)$$

obviously fulfils the boundary condition $\partial\psi/\partial z = 0$ on $z = 0$. According to (1.2) the stress-tensor component to be evaluated is

$$S_{zz} = \frac{1}{2} \left[\left(\frac{\partial \psi}{\partial t} \right)^2 - \left(\frac{\partial \psi}{\partial x} \right)^2 - \left(\frac{\partial \psi}{\partial y} \right)^2 \right].$$

Its matrix element is found to be

$$\langle \boldsymbol{k}, \boldsymbol{k}' \mid \overline{S_{zz}} \mid 0 \rangle = \frac{1}{(2\pi)^3} \frac{2}{\sqrt{\omega\omega'}} \left(-\omega\omega' + k_x k_x' + k_y k_y' \right) \times \mathrm{e}^{-(\omega+\omega')T - \mathrm{i}(\boldsymbol{k}_\parallel + \boldsymbol{k}_\parallel') \cdot \boldsymbol{r}_\parallel} \tag{A.1}$$

where Lorentzian time averaging has been already applied.

For a small piston ($a \ll T$) averaging over its surface makes no further difference (Barton 1991a) since the time averaging over times long compared with the typical geometrical dimensions of the piston leaves only the long-wavelengths modes which are anyway coherent over those short distances on the piston. Squaring the matrix element (A.1) and integrating over all modes gives

$$\Delta \overline{S_{zz}}^2 = \frac{1}{32\pi^6} \int_0^\infty \mathrm{d}\ell \int_0^\infty \mathrm{d}\ell' \int \mathrm{d}^2 \boldsymbol{k}_\parallel \int \mathrm{d}^2 \boldsymbol{k}_\parallel' \, \mathrm{e}^{-2(\omega+\omega')T} \frac{(\boldsymbol{k}_\parallel \cdot \boldsymbol{k}_\parallel' - \omega\omega')^2}{\omega\omega'}$$

and finally

$$\Delta \overline{S_{zz}}^2 = \frac{11}{2^9 \pi^4 T^8}. \tag{A.2}$$

For a large piston ($a \gg T$) the surface averaging over a square piston with side length a implies

$$\frac{1}{a^2} \int_{-a/2}^{a/2} \mathrm{d}x \int_{-a/2}^{a/2} \mathrm{d}y \, \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_\parallel + \boldsymbol{k}_\parallel') \cdot \boldsymbol{r}_\parallel} = \frac{4}{a^2} \frac{\sin[(k_x + k_x')a/2]}{k_x + k_x'} \frac{\sin[(k_y + k_y')a/2]}{k_y + k_y'}$$
$$= \frac{4\pi^2}{a^2} \delta^{(2)}(\boldsymbol{k}_\parallel + \boldsymbol{k}_\parallel').$$

† The index $\parallel$ here and in the following marks two-dimensional vectors in the (x, y) or (k_x, k_y) planes.

In the last step the formula $\sin(x/\varepsilon)/(\pi x)\xrightarrow{\varepsilon\to 0}\delta(x)$ was used, since the side length a is large compared to the wavelength. Then one has

$$\left|\frac{1}{a^2}\int_{-a/2}^{a/2}\mathrm{d}x\int_{-a/2}^{a/2}\mathrm{d}y\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_\parallel+\boldsymbol{k}'_\parallel)\cdot\boldsymbol{r}_\parallel}\right|^2 = \frac{4\pi^2}{a^2}\,\delta^{(2)}(\boldsymbol{k}_\parallel+\boldsymbol{k}'_\parallel)\,.$$

In order to avoid a squared delta function, one factor of the squared integral was evaluated first; the second integral then gives 1, since the exponent of the integrand vanishes due to the delta function in front.

Using this for surface averaging and squaring (A.1) one finds

$$\Delta\overline{S_{zz}}^2 = \frac{1}{8\pi^4a^2}\int_0^\infty \mathrm{d}\ell\int_0^\infty \mathrm{d}\ell'\int \mathrm{d}^2\boldsymbol{k}_\parallel\,\mathrm{e}^{-2(\omega+\omega')T}\omega\omega'\left(\frac{k_\parallel^2}{\omega\omega'}+1\right)^2\,.$$

After substituting ℓ and ℓ' according to $\Omega^2=(\ell^2+k_\parallel^2)/k_\parallel^2$ and performing the $\boldsymbol{k}_\parallel$ integration (in polar coordinates) one obtains

$$\begin{aligned}\Delta\overline{S_{zz}}^2 = {}& \frac{15}{2^5\pi^3a^2T^6}\int_1^\infty \mathrm{d}\Omega\int_1^\infty \mathrm{d}\Omega'\,\frac{\Omega^2\Omega'^2}{\sqrt{\Omega^2-1}\sqrt{\Omega'^2-1}}\\ &\times\left[\frac{1}{(\Omega\Omega')^2}+\frac{2}{\Omega\Omega'}+1\right]\frac{1}{(\Omega+\Omega')^6}.\end{aligned}$$

The integrals occuring here are already known from the calculations for the large spheres and given by (B.8) to (B.10). So, the final result for the large piston is

$$\Delta\overline{S_{zz}}^2 = \frac{11}{3\times 2^7\pi^3a^2T^6}\,. \tag{A.3}$$

A2. Dirichlet boundary conditions

If the boundary condition $\psi(z=0)=0$ is imposed the scalar field may be expressed by

$$\psi = \frac{1}{(2\pi)^{3/2}}\int_0^\infty \mathrm{d}\ell\int \mathrm{d}^2\boldsymbol{k}_\parallel\sqrt{\frac{2}{\omega}}\sin\ell z\left(u_k\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\parallel\cdot\boldsymbol{r}_\parallel-\mathrm{i}\omega t}+u_k^\dagger\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\parallel\cdot\boldsymbol{r}_\parallel+\mathrm{i}\omega t}\right).$$

The boundary condition reduces the stress-tensor to

$$S_{zz}=\frac{1}{2}\left(\frac{\partial\psi}{\partial z}\right)^2$$

leading to the (time-averaged) matrix element

$$\langle\boldsymbol{k},\boldsymbol{k}'\mid\overline{S_{zz}}\mid 0\rangle = \frac{1}{(2\pi)^3}\frac{2}{\sqrt{\omega\omega'}}\ell\ell'\,\mathrm{e}^{-(\omega+\omega')T-\mathrm{i}(\boldsymbol{k}_\parallel+\boldsymbol{k}'_\parallel)\cdot\boldsymbol{r}_\parallel}\,.$$

For a small piston it is easily found that

$$\Delta\overline{S_{zz}}^2 = \frac{1}{2^9\pi^4T^8}\,. \tag{A.4}$$

Along the same lines as before for Neumann boundary conditions one gets for a large piston

$$\Delta\overline{S_{zz}}^2 = \frac{1}{8\pi^4 a^2}\int_0^\infty \mathrm{d}\ell \int_0^\infty \mathrm{d}\ell' \int \mathrm{d}^2\boldsymbol{k}_\parallel\, \mathrm{e}^{-2(\omega+\omega')T}\frac{\ell^2\ell'^2}{\omega\omega'}$$
$$= \frac{15}{2^5\pi^3 a^2 T^6}\int_1^\infty \mathrm{d}\Omega \int_1^\infty \mathrm{d}\Omega'\, \frac{\sqrt{\Omega^2-1}\sqrt{\Omega'^2-1}}{(\Omega+\Omega')^6}$$

and by use of (B.7)

$$\Delta\overline{S_{zz}}^2 = \frac{1}{3\times 2^7\pi^3 a^2 T^6}\,. \tag{A.5}$$

Appendix B. Miscellanea

B1. Integration over the surface of a sphere

The integrals to be evaluated are:

$$\Im_1 = \int \mathrm{d}\Omega \left(\frac{mm'}{\sin^2\theta}\cos\theta\, Y_\ell^{m*}(\hat{\boldsymbol{r}})Y_{\ell'}^{m'*}(\hat{\boldsymbol{r}}) - \cos\theta\,\frac{\partial Y_\ell^{m*}(\hat{\boldsymbol{r}})}{\partial\theta}\frac{\partial Y_{\ell'}^{m'*}(\hat{\boldsymbol{r}})}{\partial\theta}\right) \tag{B.1}$$

$$\Im_2 = \Im_1 - \frac{\ell(\ell+1)}{kR}\frac{\ell'(\ell'+1)}{k'R}\int \mathrm{d}\Omega\cos\theta\, Y_\ell^{m*}(\hat{\boldsymbol{r}})Y_{\ell'}^{m'*}(\hat{\boldsymbol{r}}) \tag{B.2}$$

$$\Im_3 = \mathrm{i}\int \mathrm{d}\Omega\left(\frac{\partial Y_\ell^{m*}(\hat{\boldsymbol{r}})}{\partial\theta}m'\cot\theta\, Y_{\ell'}^{m'*}(\hat{\boldsymbol{r}}) - m\cot\theta\, Y_\ell^{m*}(\hat{\boldsymbol{r}})\frac{\partial Y_{\ell'}^{m'*}(\hat{\boldsymbol{r}})}{\partial\theta}\right). \tag{B.3}$$

Decomposing the angular momentum operator $\boldsymbol{L}$ into its cartesian components one easily recognizes that the above integrals may be expressed in terms of scalar products in the Hilbert space of the angular momentum eigenfunctions $Y_\ell^m(\hat{\boldsymbol{r}})$,

$$\Im_1 = (-1)^{m'+1}\langle \boldsymbol{L}Y_\ell^m \mid \cos\theta \mid \boldsymbol{L}Y_{\ell'}^{-m'}\rangle$$
$$\Im_2 = \Im_1 + \frac{\ell(\ell+1)}{kR}\frac{\ell'(\ell'+1)}{k'R}(-1)^{m'+1}\langle Y_\ell^m \mid \cos\theta \mid Y_{\ell'}^{-m'}\rangle$$
$$\Im_3 = (-1)^{m'+1}\langle Y_\ell^m \mid \mathrm{i}L_z \mid Y_{\ell'}^{-m'}\rangle.$$

The solution of the third integral is trivial now,

$$\Im_3 = (-1)^{m'}\mathrm{i}m'\,\delta_{\ell\ell'}\delta_{m-m'}. \tag{B.4}$$

The integral $\Im_1$ is solved by expressing the scalar product of the angular momentum operators in terms of the angular-momentum raising and lowering operators, L_+ and L_-. Their action on eigenstates of $\boldsymbol{L}$ is known; so it remains to evaluate the scalar product

$$\langle Y_\ell^m \mid \cos\theta \mid Y_j^{-m'}\rangle = \sqrt{\frac{4\pi}{3}}\langle Y_\ell^m \mid Y_1^0 \mid Y_j^{-m'}\rangle$$

which is, by use of the Wigner–Eckart theorem,

$$= \sqrt{\frac{4\pi}{3}}\langle j\,1\,m\,0 \mid \ell\,m\rangle\langle\ell\|Y_1\|j\rangle\,\delta_{m-m'}$$
$$= \sqrt{\frac{2j+1}{2\ell+1}}\langle j\,1\,m\,0 \mid \ell\,m\rangle\langle j\,1\,0\,0 \mid \ell\,0\rangle\,\delta_{m-m'}.$$

Insertion of the appropriate Clebsch–Gordan coefficients (AS, table 27.9.2) then gives eventually

$$\Im_1 = (-1)^{m+1}\delta_{m-m'}\left[\delta_{\ell\,\ell'+1}(\ell^2-1)\sqrt{\frac{(\ell-m)(\ell+m)}{(2\ell+1)(2\ell-1)}}\right.$$
$$\left. + \,\delta_{\ell\,\ell'-1}(\ell'^2-1)\sqrt{\frac{(\ell'-m)(\ell'+m)}{(2\ell'+1)(2\ell'-1)}}\right] \tag{B.5}$$

$$\Im_2 = (-1)^{m+1}\delta_{m-m'}\left[\delta_{\ell\,\ell'+1}(\ell^2-1)\left(1+\frac{\ell^2}{kk'R^2}\right)\sqrt{\frac{(\ell-m)(\ell+m)}{(2\ell+1)(2\ell-1)}}\right.$$
$$\left. + \,\delta_{\ell\,\ell'-1}(\ell'^2-1)\left(1+\frac{\ell'^2}{kk'R^2}\right)\sqrt{\frac{(\ell'-m)(\ell'+m)}{(2\ell'+1)(2\ell'-1)}}\right]. \tag{B.6}$$

B2. Two-dimensional integrals for the short-wavelengths limit

The integral

$$\int_1^\infty \mathrm{d}z \int_1^\infty \mathrm{d}\tilde{z}\, \frac{\sqrt{z^2-1}\,\sqrt{\tilde{z}^2-1}}{(z+\tilde{z})^6}$$

may be rewritten as

$$= \int_0^\infty \mathrm{d}\alpha \int_0^\infty d\beta\, \frac{\sinh^2\alpha \sinh^2\beta}{(\cosh\alpha+\cosh\beta)^6}$$

where the substitutions $z = \cosh\alpha$, $\tilde{z} = \cosh\beta$ were made. With the help of the standard addition formulae for hyperbolic functions (AS 4.5.38, 43) and after substituting $p = (\alpha+\beta)/2$, $q = (\alpha-\beta)/2$, one finds

$$= \frac{1}{2^5}\int_0^\infty \mathrm{d}p \int_{-p}^{p} \mathrm{d}q \frac{\left[\frac{1}{2}\cosh(2p) - \frac{1}{2}\cosh(2q)\right]^2}{\cosh^6 p\, \cosh^6 q}.$$

Now the integral factorizes and can be evaluated by standard integrations. Eventually this leads to

$$\int_1^\infty \mathrm{d}z \int_1^\infty \mathrm{d}\tilde{z}\, \frac{\sqrt{z^2-1}\sqrt{\tilde{z}^2-1}}{(z+\tilde{z})^6} = \frac{1}{180}. \tag{B.7}$$

In precisely the same way the results for the other integrals are obtained;

$$\int_1^\infty \mathrm{d}z \int_1^\infty \mathrm{d}\tilde{z}\, \frac{z^2\tilde{z}^2}{\sqrt{z^2-1}\,\sqrt{\tilde{z}^2-1}}\, \frac{1}{(z+\tilde{z})^6} = \frac{23}{900} \tag{B.8}$$

$$\int_1^\infty \mathrm{d}z \int_1^\infty \mathrm{d}\tilde{z}\, \frac{z\tilde{z}}{\sqrt{z^2-1}\,\sqrt{\tilde{z}^2-1}}\, \frac{1}{(z+\tilde{z})^6} = \frac{1}{75} \tag{B.9}$$

$$\int_1^\infty \mathrm{d}z \int_1^\infty \mathrm{d}\tilde{z}\, \frac{1}{\sqrt{z^2-1}\,\sqrt{\tilde{z}^2-1}}\, \frac{1}{(z+\tilde{z})^6} = \frac{2}{225}. \tag{B.10}$$

References

Abramowitz M and Stegun I A (eds) 1964 *Handbook of Mathematical Functions* (Washington, DC: US Govt Printing Office)
Balian R and Duplantier B 1977 *Ann. Phys. NY* 104 300
Barton G 1991a *J. Phys. A: Math. Gen.* 24 991
—— 1991b *J. Phys. A: Math. Gen.* **24** 5533
Biedenharn L C and Louck J D 1989 *Angular Momentum in Quantum Physics, Theory and Application* Section 7.6 (Cambridge: Cambridge University)
Brink D M and Satchler G R 1961 *Angular Momentum* (Oxford: Clarendon)
Casimir H B G 1948 *Proc. Kon. Ned. Akad. Wetenschap.* B 51 793
Eberlein C 1992 *J. Phys. A: Math. Gen.* **25** 3039–50
Erdélyi A, Magnus W, Oberhettinger F and Tricomi F G *Higher Transcendental Functions (Bateman Manuscript Project)* vol. 2 (New York: McGraw-Hill)
Gradshteyn I S and Ryshik I M 1980 *Tables of Integrals, Series, and Products* (New York: Academic)
Mott N F and Massey H S W 1965 *The Theory of Atomic Collisions* 3rd ed (Oxford: Clarendon)
Newton R G 1982 *Scattering Theory of Waves and Particles*, 2nd ed (New York: Springer)
Ruppin R 1982 *Electromagnetic Surface Modes* ed A D Boardman (New York: Interscience)
Shabde N G 1937 *Bull. Calcutta. Math. Soc.* 29 33
Wu T T 1956 *Phys. Rev.* 104 1201